13 Distributive Lattices:- for $a, b, c \in L$.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$\& \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Thm 1. A lattice is distributive iff
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad - (X)$$

Sol:- Let L be distributive lattice

i.e. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad - (1)$

T.P $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

R.H.S $(a \vee b) \wedge (a \vee c)$

$= \{(a \vee b) \wedge a\} \vee \{(a \vee b) \wedge c\}$ (By (1))

$= a \vee \{(a \vee b) \wedge c\}$ (By absorption law)

$\Rightarrow a \vee \{c \wedge (a \vee b)\}$

$\Rightarrow a \vee \{(c \wedge a) \vee (c \wedge b)\}$ (By (1))

$= a \vee (b \wedge c)$ (By absorption law)

$\Rightarrow$ L.H.S

Converse:- Let $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad - (2)$

T.P $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

R.H.S

$(a \wedge b) \vee (a \wedge c) = \{(a \wedge b) \vee a\} \wedge \{(a \wedge b) \vee c\}$ (By (2))

$= a \wedge \{(a \wedge b) \vee c\}$ (By absorp. law)

$= a \wedge \{c \vee (a \wedge b)\}$

$= a \wedge \{(c \vee a) \wedge (c \vee b)\}$ By (2)

$= \{a \wedge (c \vee a)\} \wedge (c \vee b)$

$= a \wedge (b \vee c)$ (By assoc. law)

$=$ L.H.S

Hence L is distributive lattice.

Thm 2. S.T. Every Chain is distributive lattice.

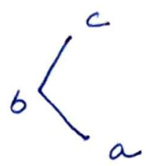
Sol :- Let $(L, \leq)$ be a chain

$a, b, c \in L$

T.P distributive law holds for $a, b, c \in L$.

Case I

$a \leq b,\ a \leq c,\ b \leq c$



L.H.S $a \wedge (b \vee c)$

$\Rightarrow a \wedge c$

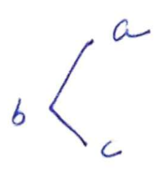
$\Rightarrow a$

R.H.S $(a \wedge b) \vee (a \wedge c)$

$a \vee a \Rightarrow a$

L.H.S = R.H.S

Case II $b \leq a,\ c \leq a,\ c \leq b$



L.H.S $a \wedge (b \vee c)$

$\Rightarrow a \wedge b$

$\Rightarrow b$

R.H.S $(a \wedge b) \vee (a \wedge c) \Rightarrow b \vee c$

$\Rightarrow b$

L.H.S = R.H.S.

Hence Every chain is distributive lattice.

Thm VII - P.T. Direct Product of Two Distributive Lattice is also distributive.

Sol:- Let L and M be two distributive lattices.

$L \times M = \{(a,b) ; a \in L, b \in M\}$

Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in L \times M$

$a_1, a_2, a_3 \in L ; b_1, b_2, b_3 \in M$.

Now L and M are distributive lattices

$\therefore \quad a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$ — ①

& $b_1 \wedge (b_2 \vee b_3) = (b_1 \wedge b_2) \vee (b_1 \wedge b_3)$ — ②

$(a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)] = (a_1, b_1) \wedge [a_2 \vee a_3, b_2 \vee b_3]$

$= [a_1 \wedge (a_2 \vee a_3), \ b_1 \wedge (b_2 \vee b_3)]$

$= [(a_1 \wedge a_2) \vee (a_1 \wedge a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3)]$

$= [(a_1 \wedge a_2, b_1 \wedge b_2) \vee (a_1 \wedge a_3), (b_1 \wedge b_3)]$

$= [(a_1, b_1) \wedge (a_2, b_2)] \vee [(a_1, b_1) \wedge (a_3, b_3)]$

Hence

$(a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)] = [(a_1, b_1) \wedge (a_2, b_2)] \vee [(a_1, b_1) \wedge (a_3, b_3)]$

Hence $L \times M$ is Distributive Lattice.

Thm IV Prove that Homomorphic image of a distributive lattice is distributive.

Sol:- Let $f : L \to M$ be onto homomorphism

L is distributive lattice

T.P M is Distributive lattice

Given f is onto

$\therefore \exists\ a, b, c \in L$

$f(a) = x,\ f(b) = y,\ f(c) = z$

Now $x \wedge (y \vee z) = f(a) \wedge (f(b) \vee f(c))$

$= f(a) \wedge (f(b \vee c))$ [$\because f$ is homomorphism]

$= f[a \wedge (b \vee c)]$

$= f[(a \wedge b) \vee (a \wedge c)]$ [$\because L$ is distributive]

$= f(a \wedge b) \vee f(a \wedge c)$ [$\because f$ is H.m]

$= [f(a) \wedge f(b)] \vee [f(a) \wedge f(c)]$

$= (x \wedge y) \vee (x \wedge z)$

$\therefore x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \ \forall\ x, y, z \in M$

Hence M is Distributive lattice

$\therefore$ Homomorphic image of a distributive lattice is distributive.